4. APPROXIMATE VALUES

In many least squares problems, the unknown quantities being sought may be quite large numbers and or the coefficients of these quantities may be large numbers. This can lead to numerical problems in the formation of normal equations where large numbers are multiplied and summed. To overcome this problem, approximate values of the unknown quantities may be used and small, unknown corrections to the approximate values become the quantities being sought.

In general, we denote unknown values as x, approximate values as x^0 and small corrections as Δx or δx and

$$x = x^0 + \delta x \tag{4.1}$$

In the case of a vector of unknown quantities $\mathbf{x}$ we have a vector of approximate values $\mathbf{x}^0$ and a vector of small corrections $\delta\mathbf{x}$ and

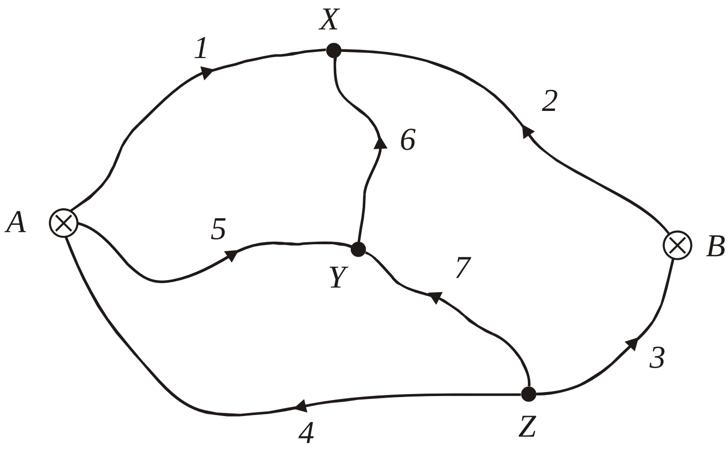
$$\mathbf{x} = \mathbf{x}^0 + \delta\mathbf{x} \tag{4.2}$$

The use of approximate values can best be explained by example and the following sections contain worked examples of some simple least squares problems that demonstrate the use of approximate values.

4.1. LEVEL NET ADJUSTMENT

The diagram below shows a level network of height differences observed between the fixed stations A (RL 102.440 m) and B (RL 104.565 m) and "floating" stations X, Y and Z whose Reduced Levels (RL's) are unknown. The arrows on the diagram indicate the direction of rise. The Table of Height differences shows the height difference for each line of the network and the distance (in kilometers) of each level run.

Line	Height Diff	Dist (km)
1	6.345	1.7
2	4.235	2.5
3	3.060	1.0
4	0.920	3.8
5	3.895	1.7
6	2.410	1.2
7	4.820	1.5



The method of Least Squares can be used to determine the best estimates of the RL's of *X*, *Y* and *Z* bearing in mind that the precision of the observed height differences is inversely proportional to the distance of the level run.

The observation equation for the RL's of two points *P* and *Q* connected by an observed spirit levelled height difference ΔH_{PQ} can be written as

$$P + \Delta H_{PQ} + v_{PQ} = Q \tag{4.3}$$

where *P* and *Q* are the RL's of points *P* and *Q* and v_{PQ} is the residual, a small unknown correction to the observed height difference. If the RL's of *P* and *Q* are unknown but have approximate values, say $P = P^0 + \delta P$ and $Q = Q^0 + \delta Q$ we may write a general observation equation for an observed height difference as

$$P^0 + \delta P + \Delta H_{PQ} + v_{PQ} = Q^0 + \delta Q \tag{4.4}$$

Using this general observation equation we may write an equation for each observed height difference

$$\begin{aligned}
A \qquad &+ \Delta H_1 + v_1 = X^0 + \delta X \\
B \qquad &+ \Delta H_2 + v_2 = X^0 + \delta X \\
Z^0 + \delta Z &+ \Delta H_3 + v_3 = B \\
Z^0 + \delta Z &+ \Delta H_4 + v_4 = A \\
A \qquad &+ \Delta H_5 + v_5 = Y^0 + \delta Y \\
Y^0 + \delta Y &+ \Delta H_6 + v_6 = X^0 + \delta X \\
Z^0 + \delta Z &+ \Delta H_7 + v_7 = Y^0 + \delta Y
\end{aligned}$$

Rearranging these equations so that all the unknown quantities are on the left-hand-side of the equals sign and all the known quantities are on the right-hand-side gives

$$
\begin{aligned}
&v_1 - \delta X && = X^0 - A \quad -\Delta H_1 \\
&v_2 - \delta X && = X^0 - B \quad -\Delta H_2 \\
&v_3 \qquad\qquad +\delta Z && = B - Z^0 \quad -\Delta H_3 \\
&v_4 \qquad\qquad +\delta Z && = A - Z^0 \quad -\Delta H_4 \\
&v_5 \qquad -\delta Y && = Y^0 - A \quad -\Delta H_5 \\
&v_6 - \delta X + \delta Y && = X^0 - Y^0 \ -\Delta H_6 \\
&v_7 \qquad -\delta Y + \delta Z && = Y^0 - Z^0 \ -\Delta H_7
\end{aligned}
$$

The approximate RL's of the unknown points *X, Y* and *Z* can be determined from the RL's of *A* and *B* and appropriate height differences

$$
\begin{aligned}
X^0 &= A + \Delta H_1 = 108.785 \text{ m} \\
Y^0 &= A + \Delta H_5 = 106.335 \text{ m} \\
Z^0 &= A - \Delta H_4 = 101.520 \text{ m}
\end{aligned}
$$

Writing these equations in the standard form $\mathbf{v} + \mathbf{Bx} = \mathbf{f}$ gives

$$
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \delta X \\ \delta Y \\ \delta Z \end{bmatrix} = \begin{bmatrix} \left(X^0 - A\right) - \Delta H_1 \\ \left(X^0 - B\right) - \Delta H_2 \\ \left(B - Z^0\right) - \Delta H_3 \\ \left(A - Z^0\right) - \Delta H_4 \\ \left(Y^0 - A\right) - \Delta H_5 \\ \left(X^0 - Y^0\right) - \Delta H_6 \\ \left(Y^0 - Z^0\right) - \Delta H_7 \end{bmatrix} = \begin{bmatrix} 0.000 \\ -0.015 \\ -0.015 \\ 0.000 \\ 0.000 \\ 0.040 \\ -0.005 \end{bmatrix}
$$

The weight matrix for the adjustment is

$$
\begin{aligned}
\mathbf{W} &= \text{diag}\begin{bmatrix} \frac{1}{1.7} & \frac{1}{2.5} & \frac{1}{1} & \frac{1}{3.8} & \frac{1}{1.7} & \frac{1}{1.2} & \frac{1}{1.5} \end{bmatrix} \\
&= \text{diag}\begin{bmatrix} 0.5882 & 0.4000 & 1.0000 & 0.2632 & 0.5882 & 0.8333 & 0.6667 \end{bmatrix}
\end{aligned}
$$

The least squares solution for the vector of corrections **x** can be obtained from the MATLAB function *least_squares.m* with the following data file `c:\Temp\Level_Net_Data.dat`

```
% Data file for Level Net Adjustment
%
%  dX   dY   dZ    f     weight
   -1   0    0    0.000  0.5882
   -1   0    0   -0.015  0.4000
    0   0    1   -0.015  1.0000
    0   0    1    0.000  0.2632
    0  -1    0    0.000  0.5882
   -1   1    0    0.040  0.8333
    0  -1    1   -0.005  0.6667
```

Running the program from the MATLAB command window created the following output file `c:\Temp|Level_Net_Data.out`

```
Least Squares Adjustment of Indirect Observations

Input Data

Coefficient matrix B of observation equations v + Bx = f
   -1.0000    0.0000    0.0000
   -1.0000    0.0000    0.0000
    0.0000    0.0000    1.0000
    0.0000    0.0000    1.0000
    0.0000   -1.0000    0.0000
   -1.0000    1.0000    0.0000
    0.0000   -1.0000    1.0000

Vector of numeric terms f and weights w of observation equations v + Bx = f
    0.0000      0.5882
   -0.0150      0.4000
   -0.0150      1.0000
    0.0000      0.2632
    0.0000      0.5882
    0.0400      0.8333
   -0.0050      0.6667

Coefficient matrix N of Normal equations Nx = t
      1.8215     -0.8333      0.0000
     -0.8333      2.0882     -0.6667
      0.0000     -0.6667      1.9299

Vector of numeric terms t of Normal equations Nx = t
   -0.0273
    0.0367
   -0.0183

Inverse of Normal equation coefficient matrix
     6.9073e-001     3.0981e-001     1.0703e-001
     3.0981e-001     6.7720e-001     2.3394e-001
     1.0703e-001     2.3394e-001     5.9898e-001

Vector of solutions x
   -0.0095
    0.0121
   -0.0053
```

```
Vector of residuals v
   -0.0095
   -0.0245
   -0.0097
    0.0053
    0.0121
    0.0184
    0.0124
```

The adjusted RL's of *X*, *Y* and *Z* are

$$\begin{aligned} X &= X^0 + \delta X = 108.785 - 0.0095 = 108.776 \text{ m} \\ Y &= Y^0 + \delta Y = 106.335 + 0.0121 = 106.347 \text{ m} \\ Z &= Z^0 + \delta Z = 101.520 - 0.0053 = 101.515 \text{ m} \end{aligned}$$

The adjusted height differences are

Line	Observed ΔH	Residual v	Adjusted ΔH
1	6.345	-0.0095	6.336
2	4.235	-0.0245	4.211
3	3.060	-0.0097	3.050
4	0.920	0.0053	0.925
5	3.895	0.0121	3.907
6	2.410	0.0184	2.428
7	4.820	0.0124	4.832